Function Theory of a Complex Variable (E2): Exercise sheet 1

1. For $z = x + iy$ (with $x, y \in \mathbb{R}$), find the real and imaginary parts of $\frac{z-1}{z+1}$.

2. Prove that
$$\left| \frac{z - w}{1 - z\bar{w}} \right| = 1$$
if either $|z| = 1$ or $|w| = 1$. What exception must be made if $|z| = |w| = 1$?

3. Check the *parallelogram law*: for $z, w \in \mathbb{C}$,
$$|z - w|^2 + |z + w|^2 = 2\left(|z|^2 + |w|^2\right).$$

4. Prove Lagrange's identity in complex form: for $z_i, w_i \in \mathbb{C}$, $i = 1, \dots, n$,
$$\left|\sum_{i=1}^{n} z_i w_i\right|^2 = \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2 - \sum_{1 \le i < j \le n} |z_i \bar{w}_j - z_j \bar{w}_i|^2.$$

5. Calculate the cube roots of i.

6. Describe geometrically the sets of points z in the complex plane defined by the following relations:
   (a) $|z - z_1| = |z - z_2|$, where $z_1, z_2 \in \mathbb{C}$.
   (b) $1/z = \bar{z}$.
   (c) $\mathrm{Re}(z) > c$, where $c \in \mathbb{R}$.
   (d) $|z| = \mathrm{Re}(z) + 1$.

7. For $z = x + iy$, $w = u + iv$ (where $x, y, u, v \in \mathbb{R}$), define the $\mathbb{R}^2$ inner product by $\langle z, w\rangle = xu + yv$, and the Hermitian inner product by $(z, w) = z\bar{w}$. Show that
$$\langle z, w\rangle = \frac{(z, w) + (w, z)}{2} = \mathrm{Re}((z, w)).$$

8. Let $x \in \mathbb{R}_+$ be such that (x, x, x) lies on the Riemann sphere. What are the absolute values of $\pi(x, x, x)$ and $\pi(x, x, -x)$, where π is the stereographic projection?

9. Recall that if $\pi^{-1}(z) = (x_1, x_2, x_3)$ and $\pi^{-1}(z') = (x'_1, x'_2, x'_3)$ for $z, z' \in \mathbb{C}_\infty$, where π is the stereographic projection, then we define the spherical distance between z and z' by setting
$$d(z, z') = \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}.$$
Check that for $z, z' \in \mathbb{C}$ this can alternatively be written
$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}.$$
Moreover, show that if $z \in \mathbb{C}$ and $z' = \infty$, then
$$d(z, z') = \frac{2}{\sqrt{1 + |z|^2}}.$$

10. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$ (where the latter equality should be considered to hold for $\{z, z'\} = \{0, \infty\}$).